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Ising ferromagnet in fractal lattices: analytical verification of the hyperscaling law

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Abstract. We use an exact recursion procedure to prove analytically the hyperscaling law extended to fractals ($d_f \nu = 2 - \alpha$) for the zero-field Ising ferromagnet in the whole family of Migdal–Kadanoff-like hierarchical lattices.

1. Introduction

Classical spin systems defined on hierarchical lattices have attracted much attention in the study of critical phenomena because these models, such as the Ising and Potts models, are exactly soluble for these lattices. These exact results on non-Bravais lattices are important tools for the knowledge of many complicated points in the phase diagrams of several models. The analytical solutions of spin models on the Bethe lattice, for example, are cited by Baxter [1] as one of the most interesting exact solutions for higher-dimensional spin systems. The Bethe lattice can be viewed as a kind of hierarchical lattice (HL) [2–5] which is a relevant family of non-Bravais lattices that can be considered, in many situations, as approximated lattices of some Bravais ones (see, e.g. [6]). This is particularly true when the Migdal–Kadanoff approximation [7, 8] is used for spin systems on hypercubic lattices, which results in the same spin systems on HLs, giving predictions in good agreement with those obtained for the corresponding Bravais lattices by other methods such as series, numerical, etc [8–13]. Some results are relatively simple to obtain using this kind of fractal lattices, in particular, critical frontiers and correlation length critical exponents. However, the exact calculation of other physical quantities, such as specific heat, magnetization, and susceptibility, as well as their corresponding critical exponents, are much more complicated to obtain within the HL approach and we sometimes find in the literature the use of heuristic recipes to obtain these functions and exponents [12, 14–16]. Concerning the critical exponents and scaling laws, the hyperscaling law ($d_f \nu = 2 - \alpha$) has been numerically verified in a number of HL systems [17–20] and has been proved analytically for the three-state antiferromagnetic Potts model on a diamond-type HL family [21]. Concerning the Rushbrooke scaling law ($\alpha + 2\beta + \gamma = 2$), there is much less evidence in favour of its validaty on fractal systems. It has been verified [22] in the Potts ferromagnet on the Wheatstone-bridge HL using approximate methods in the derivation of β and γ. It has also been verified for the Ising ferromagnet in an m-sheet Sierpinski gasket family using numerical values for the exponents derived from exact expressions of the thermal quantities [19].

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Here we use a method that allows us to calculate exactly the exponents ν and α of the Ising ferromagnet on the family of Migdal–Kadanoff-type HLs. This method was used to calculate exactly several thermodynamical functions and critical exponents of the Ising model on HLs [23], the q-state Potts model on some diamond-type HLs [20] and the $q = 3$ Potts antiferromagnet on a family of diamond-type HLs [21]. With the critical exponents we obtained, we verify analytically, without any intermediary numerical calculation, for the first time, as far as we know, the hyperscaling law for the Ising ferromagnet on a large family of fractal lattices. The outline of this paper is as follows. In section 2 we define the model on the HLs and explain the method we use. In section 3 we derive a recursion relation for the systems' internal energy and the renormalization group (RG) transformation of the ferromagnetic coupling. In section 4 we verify analytically the hyperscaling law and give conclusions in section 5.

2. Model

We consider the zero-field Ising ferromagnet on the family of the Migdal–Kadanoff-like HLs. These lattices are generated in an iterative manner, starting from a two-point lattice joined by a single bond (level $n = 0$) which is replaced by a basic cell consisting of P branches in parallel, each of them comprising b bonds in series. The n level is obtained from the previous one by replacing each bond by the basic cell. This recursive procedure is illustrated in figure 1 for the cases ($P = 2$, $b = 2$) (the diamond HL) and ($P = 2$, $b = 3$).

In the $n \to \infty$ limit one obtains a lattice, which we denote as $\mathrm{HL}_{(P,b)}$, with fractal dimension

$$d_f(P, b) = \frac{\ln Pb}{\ln b}. \tag{1}$$

In the following calculations the paramaters $P \geqslant 2$ and $b \geqslant 2$ are fixed. The notations $F_{(P,b)}$ or $F^{(P,b)}$ will denote the quantity F calculated for the $\mathrm{HL}_{(P,b)}$ particular case of the HL family.

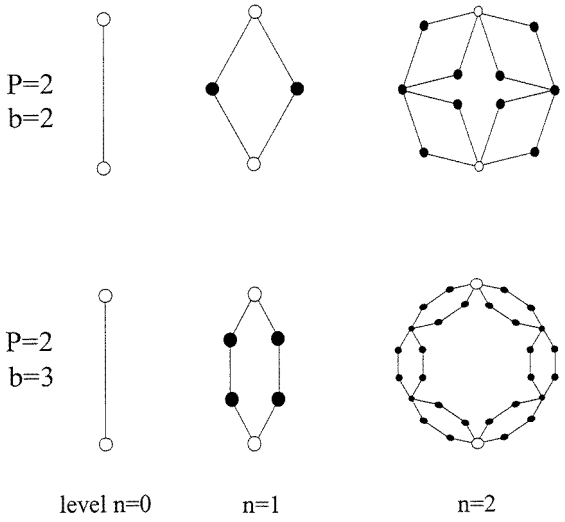

Figure 1. The first three steps of construction of the $\mathrm{HL}_{(P=2,b=2)}$ and $\mathrm{HL}_{(P=2,b=3)}$. The open circles are the root sites of the HLs.

At each vertex i of the $\mathrm{HL}_{(P,b)}$ at the n level we associate an Ising spin variable $\sigma_i = \pm 1$, and consider the model described by the dimensionless Hamiltonian

$$-\beta\mathcal{H}_n = K_n \sum_{\langle i,j\rangle} \sigma_i \sigma_j \tag{2}$$

where $\beta = 1/k_B T$, T being the temperature, $K_n = \beta J_n$ and $J_n > 0$ is the coupling constant between nearest-neighbour pairs at the n level. The sum is over all the first neighbours $\langle i, j\rangle$ of the lattice. The partition function, at the nth level, can be formally calculated through the expression

$$Z_n = \sum_{\{\sigma\}} \exp\left(K_n \sum_{\langle i,j\rangle} \sigma_i \sigma_j\right). \tag{3}$$

If we perform a partial trace over all the spins at the nth level HL, with the exception of those spins belonging to a particular basic cell (see figure 2), this partition function will be expressed as a trace over the remaining spins (those belonging to the basic cell) with an equivalent Hamiltonian $\mathcal{H}_E$ given by (for the HL shown in figure 2)

$$\mathcal{H}_E = C + K_n(\mu_1\sigma + \mu_1\sigma' + \mu_2\sigma + \mu_2\sigma') + K_E\mu_1\mu_2 + H_1\mu_1 + H_2\mu_2 \tag{4}$$

where C, K_E, H_1 and H_2 are unknown functions of the coupling constant K_n. The chosen basic cell is connected with the rest of the lattice by the spins μ_1 and μ_2. σ and σ' are the internal spins of the basic cell which appear only at the nth level of the lattice. The effective fields (H_1 and H_2) and coupling (K_E) are due to the influence of the rest of the lattice over the external spins μ_1 and μ_2. In the next section we use this effective Hamiltonian to calculate the relation between the internal energies at different levels of the HL.

3. Recursion relations

The dimensionless internal energy per bond for the n-level system is given by

$$E_n = \frac{\langle \mathcal{H}_n \rangle}{-J_n N_{(P,b)n}} = \langle \mu\sigma \rangle \tag{5}$$

where $N_{(P,b)n}$ is the number of bonds and μ and σ are any two nearest-neighbour spins at the n-level $\mathrm{HL}_{(P,b)}$.

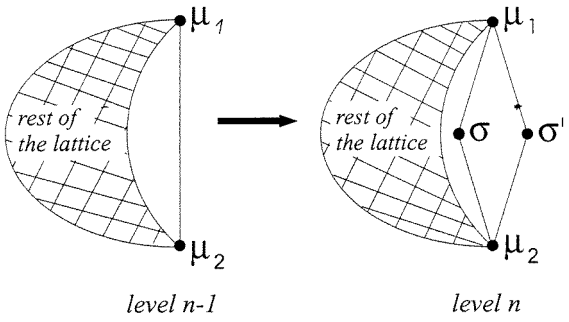

Figure 2. At the level n, each bond of the $(n-1)$-level lattice is replaced by the basic cell (with $P = 2$ and $b = 2$ in this figure). The averages involving the sites which appear only at the nth level (σ and σ') are related to the averages involving the sites of the previous level (μ_1 and μ_2) through the system of equations derived from the equivalent Hamiltonian given in the text.

Using the Hamiltonian given by equation (4) (and its generalizations for other lattices) we can calculate the quantities related to the spins μ_1 and μ_2 which can be formally expressed as

$$\begin{aligned} Z_n &= f_1(K_n, K_E, H_1, H_2, C) \\ Z_n\langle\mu_1\mu_2\rangle &= f_2(K_n, K_E, H_1, H_2, C) \\ Z_n\langle\mu_1\rangle &= f_3(K_n, K_E, H_1, H_2, C) \\ Z_n\langle\mu_2\rangle &= f_4(K_n, K_E, H_1, H_2, C). \end{aligned} \tag{6}$$

We can also calculate the averages involving the spins belonging to the latest level (n), in particular

$$\langle\sigma_1\sigma_2\rangle = g_1(K_n, K_E, H_1, H_2, C) \tag{7}$$

where σ_1 and σ_2 are first-neighbour spins in the $\mathrm{HL}_{(P,b)}$, and thus, from equation (5), $\langle\sigma_1\sigma_2\rangle = E_n$. Inverting the system given by equations (6), we obtain the parameters C, K_E, H_1 and H_2 as functions of the averages involving spins μ of the level $n-1$ which can be substituted in equation (7) to obtain an exact recursion relation for the dimensionless energy E_n. This recursion can be formally expressed as

$$E_n^{(b)} = \frac{A_n^{(b)}}{B_n^{(b)}} E_{n-1}^{(b)} + \frac{C_n^{(b)}}{B_n^{(b)}} \tag{8}$$

where $A_n^{(b)}$, $B_n^{(b)}$ and $C_n^{(b)}$, which depend on the particular $\mathrm{HL}_{(P,b)}$ lattice we use through the parameter b (the parameter P does not appear due to the equivalence between the parallel branches in the basic cell), are functions of the coupling K_n. Therefore, since we fix the system energy at the zeroth level (a single bond), it is straightforward to calculate the energy at the successive levels by the iteration of equation (8). The explicit expressions of the functions $A_n^{(b)}$, $B_n^{(b)}$ and $C_n^{(b)}$ can be easily obtained by algebraic computation. We obtain

$$\begin{aligned} A^{(b)} &= (x^4-1)^{b-1} \qquad \text{for } b = 2, 3, \ldots \\ C^{(2)} &= x^4 - 1 \\ C^{(b)} &= (x^4-1)B^{(b-1)} \qquad \text{for } b = 3, 4, \ldots \end{aligned}$$

and

$$\begin{aligned} B^{(2)} &= 2(x^4+1) \\ B^{(3)} &= (x^4+3)(3x^4+1) \\ B^{(4)} &= 4(x^4+1)(x^8+6x^4+1) \\ B^{(5)} &= (x^8+10x^4+5)(5x^8+10x^4+1) \\ B^{(6)} &= 2(x^4+3)(x^4+1)(3x^4+1)(x^4-2x^3+2x^2+2x+1)(x^4+2x^3+2x^2-2x+1) \\ &\ldots \\ B^{(10)} &= 2(x^4+1)(5x^8+10x^4+1)(x^8+10x^4+5)(x^{16}+44x^{12}+166x^8+44x^4+1) \\ &\ldots \end{aligned} \tag{9}$$

where we defined the variable $x \equiv \mathrm{e}^K$.

In order to complete our recursive equations we need the renormalization of the coupling $K_{n-1} = K'$ (or, equivalently $x_{n-1} = x'$) in terms of the coupling $K_n = K$ (or $x_n = x$). This is established in a standard way by preserving the correlation function between the roots of

the HL, i.e. by explicitly computing the partial trace over the internal spins of the $HL_{(P,b)}$s basic cell and imposing the following equality

$$\mathrm{Tr}_{\text{internal spins}} \exp \mathcal{H}_1(K) = \exp(\mathcal{C} + \mathcal{H}_0(K')) \tag{10}$$

where $\mathcal{H}_1$ is the Hamiltonian of the basic cell (level $n = 1$), $\mathcal{H}_0$ is the Hamiltonian of one single bond (level $n = 0$) and $\mathcal{C}$ is a constant. We obtain

$$x'(x) = \left(\frac{1 + U(x)^b}{1 - U(x)^b}\right)^{P/2} \qquad \text{where } U(x) \equiv \frac{x^2 - 1}{x^2 + 1}. \tag{11}$$

For $b = 2$ (the diamond-type HLs) our renormalization group (RG) transformation recovers the result $K' = P \ln \sqrt{\cosh(2K)}$ of [24]. Equation (11) admits, for all $P \geqslant 2$ and $b \geqslant 2$, two trivial stable fixed points, namely $x = 1$ ($T \to \infty$) (paramagnetic phase) and $x \to \infty$ ($T = 0$) (ferromagnetic phase), as well as a critical (unstable) fixed point denoted by $x^*_{(P,b)}$ ($0 < x^*_{(P,b)} < \infty$). Linearization of equation (11) in the neighbourhood of the critical point $x^*_{(P,b)}$ leads to the corresponding thermal (correlation length) critical exponent $\nu_{(P,b)}$:

$$\nu_{(P,b)} = \frac{\ln b}{\ln r_{(P,b)}} \qquad \text{where } r_{(P,b)} \equiv \left.\frac{\mathrm{d}x'(x)}{\mathrm{d}x}\right|_{x^*_{(P,b)}}. \tag{12}$$

4. Hyperscaling law

With the above recursive relations, the hyperscaling law extended to fractal systems, namely

$$d_f \nu = 2 - \alpha \tag{13}$$

can be proved analytically for the Ising ferromagnet for the whole set of $HL_{(P,b)}$ fractals. First, assuming in the neighbourhood of the critical temperature $T_{c(P,b)}$ (or $x^*_{(P,b)}$) that the energy E_n can be written as $E_n = E_c + \lambda(\epsilon_n)^\sigma$, where $E_c = E(x^*_{(P,b)})$ and $\epsilon_n = (x^*_{(P,b)} - x_n)$, we obtain, from equation (8), the following expression for the exponent σ:

$$\sigma_{(P,b)} = \frac{\ln(B^{(b)}(x)/A^{(b)}(x)) \mid_{x^*_{(P,b)}}}{\ln r_{(P,b)}}. \tag{14}$$

Since the specific heat is related to the derivative of the internal energy with respect to the temperature, we obtain that $\alpha_{(P,b)} = 1 - \sigma_{(P,b)}$.

Using the expressions of equations (1), (12) and (14) in equation (13) we obtain

$$bP = r_{(P,b)} \left.\frac{B^{(b)}}{A^{(b)}}\right|_{x^*_{(P,b)}} \tag{15}$$

which, after some simplifications, reduces to the final form of the hyperscaling law, in our notation, for the $HL_{(P,b)}$:

$$\frac{1}{4} = \frac{x_c^2}{(x_c^4 - 1)} \frac{1}{(U_c^{-b} - U_c^b)} \left.\frac{B^{(b)}}{A^{(b)}}\right|_{x_c} \tag{16}$$

where $x_c \equiv x^*_{(P,b)}$, U is defined in equation (11), and $U_c = U(x_c)$. This identity is exactly verified for all the $HL_{(P,b)}$ fractals we tested, in particular, for the cases given explicitly in equation (10). In the simpler case ($b = 2$), for instance, the right-hand side of equation (16) is given by:

$$\left.\frac{B^{(2)}}{A^{(2)}}\right|_{x_c} = \frac{2(x_c^4 + 1)}{x_c^4 - 1} \tag{17}$$

while

$$\frac{1}{(U_c^{-2} - U_c^2)} = \frac{1}{8}\frac{(x_c^4 - 1)^2}{x_c^2(x_c^4 + 1)} \tag{18}$$

and thus

$$\frac{B^{(2)}}{A^{(2)}}\frac{1}{(U_c^{-2} - U_c^2)} = \frac{1}{4}\frac{x_c^4 - 1}{x_c^2} \tag{19}$$

which, without any numerical calculation, gives the expected result.

5. Conclusions

We used a method that allowed us to calculate exact recurrence relations for the energy of the zero-field Ising ferromagnet for the whole family of Migdal–Kadanoff-like hierarchical lattices and to obtain the critical exponents ν and α. With the exact expressions of these exponents we proved analytically, without any intermediary numerical calculation, the validity of the hyperscaling law for this large class of fractal systems.

As is well known, the hyperscaling is an immediate outcome of the RG in the absence of dangerous irrelevant variables. However, the point of this paper was to provide an explicit, analytical verification of this law, something that has not been done explicitly earlier for this large class of systems. Furthermore, as a final comment, if we admit, *a priori*, the exact validity of the hyperscaling law for these systems (since the RG is exact) the functions $B^{(b)}(x)$ which appear in equation (9) have the general form in terms of the lattice parameter b:

$$B^{(b)}(x) = \frac{1}{4x^2}\{(x^2 + 1)^{2b} - (x^2 - 1)^{2b}\} \tag{20}$$

a formula that, I believe, cannot be deduced by any other independent method.

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